Computer Controlled Systems Lecture Notes

Vághy Mihály

Contents

1 Basic notions

1.1 Signal

Scalar- or vector-valued time-dependent functions, which describe interactions between objects in the real world with time-depentent behavior are called signals. Given a vector-valued signal

$$x : \mathbb{R} \mapsto \mathbb{R}^n$$

we can give the value of the signal at any given time t, as a vector $x(t)$.

1.2 Signal space

The set of all possible time-dependent functions which can be realizations of a signal form a signal space $\mathbb{X}$.

1.3 System

A system is part of the real world with a boundary between the system and its environments, through which the system interacts with its environment. The effects of the environment are described by a time-dependent function $u(t) \in \mathbb{U}$. The effects of the system are described by a time-dependent function $y(t) \in \mathbb{Y}$.

1.4 Special properties of systems

1. A system $\mathcal{S}$ is called linear if

$$\mathcal{S}[c_1 u_1 + c_2 u_2] = c_1 \mathcal{S}[u_1] + c_2 \mathcal{S}[u_2] = c_1 y_1 + c_2 y_2$$

   with $c_1, c_2 \in \mathbb{R}$, $u_1, u_2 \in \mathbb{U}$, and $y_1, y_2 \in \mathbb{Y}$.

2. A system $\mathcal{S}$ is time-invariant if

$$\mathcal{T}(\tau) \circ \mathcal{S} = \mathcal{S} \circ \mathcal{T}(\tau)$$

   where $\mathcal{T}$ is called time shift operator and

$$\mathcal{T}(\tau)[u(t)] = u(t + \tau).$$

3. In continuous time systems the time variable is $t \in \mathbb{T} \subset \mathbb{R}$.
4. In discrete time systems the time variable is $t \in \mathbb{T} = \{ \dots, t_0, t_1, t_2, \dots \}$.
5. Single-input single-output or SISO system.
6. Multiple-input multiple-output or MIMO system.
7. A system is called causal if the signals and the system operator are independent from the future.

2 Analysis of Continuous Time LTI Systems

2.1 Time domain description

1. Linear differential equations with constant coefficients

$$a_n \frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots + a_1 \frac{\mathrm{d}y}{\mathrm{d}t} + a_0 y = b_0 u + b_1 \frac{\mathrm{d}u}{\mathrm{d}t} + \cdots + b_{n-1}\frac{\mathrm{d}^{n-1} u}{\mathrm{d}t^{n-1}} + b_n \frac{\mathrm{d}^n u}{\mathrm{d}t^n}$$

with given initial conditions

$$y(0) = y_{0,0} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = y_{1,0} \quad \ldots \quad \frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} = y_{n-1,0}.$$

2. Impulse response representation
The output of $\mathcal{S}$ can be written as

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)\,\mathrm{d}\tau = (h * u)(t)$$

where $h(t)$ is the impulse response function, the response of the system to the Dirac-delta function $\delta(t)$. In causal systems this can be rewritten as

$$y(t) = \int_0^t h(t-\tau)u(\tau)\,\mathrm{d}\tau = \int_0^{\infty} h(\tau)u(t-\tau)\,\mathrm{d}\tau\,.$$

2.2 Laplace-transform

If $f(t)$ can be integrated absolutely and $\exists k$ such that $\lim_{t\to\infty} k f(t) e^{-\alpha t} = 0$, then the L-transform of $f(t)$ is

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}\,\mathrm{d}t\,.$$

2.3 Inverse Laplace-tranform

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}\,\mathrm{d}s\,.$$

2.4 Properties of the L-transform

1. $$\mathcal{L}\{c_1 y_1 + c_2 y_2\} = c_1\mathcal{L}\{y_1\} + c_2\mathcal{L}\{y_2\}$$

2. $$\mathcal{L}\big\{(h*u)(t)\big\} = H(s)U(s)$$

3. $$\mathcal{L}\left\{\frac{dy}{\mathrm{d}t}\right\} = \mathcal{L}\big\{\dot{y}(t)\big\} = s\mathcal{L}\{y(t)\} = sY(s) - y(0)$$

4. $$\mathcal{L}\big\{\ddot{y}(t)\big\} = s^2 Y(s) - sy(0) - \dot{y}(0)$$

5. $$y(0) = \lim_{s\to\infty} sY(s)$$

6. $$y(\infty) = \lim_{s\to 0} sY(s)$$

Proof

1.
$$\int_0^\infty (c_1 y_1 + c_2 y_2) e^{-st}\,\mathrm{d}t = \int_0^\infty \left(c_1 y_1 e^{-st} + c_2 y_2 e^{-st}\right)\mathrm{d}t = c_1 \int_0^\infty y_1 e^{-st}\,\mathrm{d}t + c_2 \int_0^\infty y_2 e^{-st}\,\mathrm{d}t$$

2.
$$\int_0^\infty \int_0^t h(\tau) u(t-\tau)\,\mathrm{d}\tau\, e^{-st}\,\mathrm{d}t = \int_0^\infty \int_0^\infty h(\tau) u(t-\tau) e^{-st}\,\mathrm{d}t\,\mathrm{d}\tau =$$
$$= \int_0^\infty \int_0^\infty h(\tau) e^{-s\tau}\,\mathrm{d}\tau\, u(t-\tau) e^{-s(t-\tau)}\,\mathrm{d}t = \int_0^\infty \int_0^\infty h(\tau) e^{-s\tau}\,\mathrm{d}\tau\, u(\vartheta) e^{-s\vartheta} d\vartheta =$$
$$= \int_0^\infty h(\tau) e^{-s\tau}\,\mathrm{d}\tau \int_0^\infty u(\vartheta) e^{-s\vartheta} d\vartheta$$

3.
$$\int_0^\infty \dot{y}(t) e^{-st}\,\mathrm{d}t = y(t) e^{-st}\Big|_0^\infty + s\int_0^\infty y(t) e^{-st}\,\mathrm{d}t = sY(s) - y(0)$$

4.
$$\int_0^\infty \ddot{y}(t) e^{-st}\,\mathrm{d}t = s\int_0^\infty \dot{y}(t) e^{-st} - \dot{y}(0) = s^2 Y(s) - sy(0) - \dot{y}(0)$$

5. We know that $\int_0^\infty \dot{y}(t) e^{-st}\,\mathrm{d}t = sY(s) - y(0)$. Then
$$\lim_{s\to\infty}\left(\int_0^\infty \dot{y}(t) e^{-st}\,\mathrm{d}t + y(0)\right) = \lim_{s\to\infty} sY(s).$$
Since
$$\lim_{s\to\infty}\int_0^\infty \dot{y}(t) e^{-st}\,\mathrm{d}t = 0$$
we get
$$y(0) = \lim_{s\to\infty} sY(s).$$

6. We know that $\int_0^\infty \dot{y}(t) e^{-st}\,\mathrm{d}t = sY(s) - y(0)$. Then
$$\lim_{s\to 0}\int_0^\infty \dot{y}(t) e^{-st}\,\mathrm{d}t = \lim_{s\to 0}\int_0^\infty \dot{y}(t)\,\mathrm{d}t = y(\infty) - y(0) = \lim_{s\to 0}\big(sY(s) - y(0)\big) = \lim_{s\to 0} sY(s) - y(0).$$
Adding $y(0)$ to both sides we get
$$y(\infty) = \lim_{s\to 0} sY(s).$$

2.5 L-transform of special functions

1.
$$\mathcal{L}\{\delta(t)\} = 1$$

2.
$$\mathcal{L}\{1(t)\} = \frac{1}{s}$$

3.
$$\mathcal{L}\{t1(t)\} = \frac{1}{s^2}$$

4.
$$\mathcal{L}\big\{e^{-at}\big\} = \frac{1}{s+a}$$

5.
$$\mathcal{L}\left\{e^{-\frac{t}{T}}\right\} = \frac{1}{s+\frac{1}{T}} = \frac{T}{1+sT}$$

6.
$$\mathcal{L}\left\{1-e^{-\frac{t}{T}}\right\} = \frac{1}{s(1+sT)}$$

7.
$$\mathcal{L}\left\{\frac{1}{T_1-T_2}\left(e^{-\frac{t}{T_1}}-e^{-\frac{t}{T_2}}\right)\right\} = \frac{1}{(1+sT_1)(1+sT_2)}$$

Proof

1.
$$\int_0^\infty \delta(t)e^{-st}\,\mathrm{d}t = \lim_{T\to\infty}\frac{1}{T}\int_0^T e^{-st}\,\mathrm{d}t = \frac{1}{s}\lim_{T\to\infty}\frac{1-e^{-sT}}{T} = 1$$

2.
$$\int_0^\infty 1(t)e^{-st}\,\mathrm{d}t = -\frac{e^{-st}}{s}\bigg|_0^\infty = \frac{1}{s}$$

3.
$$\int_0^\infty t1(t)e^{-st}\,\mathrm{d}t = \int_0^\infty te^{-st}\,\mathrm{d}t = -\frac{te^{-st}}{s}\bigg|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,\mathrm{d}t = -\frac{e^{-st}}{s^2}\bigg|_0^\infty = \frac{1}{s^2}$$

4.
$$\int_0^\infty e^{-at}e^{-st}\,\mathrm{d}t = \int_0^\infty e^{-(s+a)t}\,\mathrm{d}t = -\frac{e^{-(s+a)t}}{s+a}\bigg|_0^\infty = \frac{1}{s+a}$$

5.
$$\int_0^\infty e^{-\frac{t}{T}}e^{-st}\,\mathrm{d}t = \int_0^\infty e^{-\left(s+\frac{1}{T}\right)t}\,\mathrm{d}t = -\frac{e^{-\left(s+\frac{1}{T}\right)t}}{s+\frac{1}{T}}\bigg|_0^\infty = \frac{1}{s+\frac{1}{T}} = \frac{T}{1+sT}$$

6.
$$\int_0^\infty \left(1-e^{-\frac{t}{T}}\right)e^{-st}\,\mathrm{d}t = \int_0^\infty e^{-st}\,\mathrm{d}t - \int_0^\infty e^{-\frac{t}{T}}\,\mathrm{d}t = \frac{1}{s} - \frac{T}{1+sT} = \frac{1}{s(1+sT)}$$

7.
$$\int_0^\infty \frac{1}{T_1-T_2}\left(e^{-\frac{t}{T_1}}-e^{-\frac{t}{T_2}}\right)e^{-st}\,\mathrm{d}t = \frac{1}{T_1-T_2}\left(\int_0^\infty e^{-\frac{t}{T_1}}e^{-st}\,\mathrm{d}t - \int_0^\infty e^{-\frac{t}{T_2}}e^{-st}\,\mathrm{d}t\right) =$$
$$= \frac{1}{T_1-T-2}\left(\frac{T_1}{1+sT_1}-\frac{T_2}{1+sT_2}\right) = \frac{1}{(1+sT_1)(1+sT_2)}$$

2.6 Operator domain description

Using the transfer function
$$H(s) = \mathcal{L}\{h(t)\} = \frac{b(s)}{a(s)}$$
(where $b(s), a(s)$ are polynomials) the description of the LTI system is
$$Y(s) = H(s)U(s)$$
where $Y(s) = \mathcal{L}\{y(t)\}$ and $U(s) = \mathcal{L}\{u(t)\}$.

2.7 Proper transfer function

Let the transfer function of a CT SISO LTI system be $H(s) = \frac{b(s)}{a(s)}$. If

1. $\deg\big(a(s)\big) > \deg\big(b(a)\big)$ then $H(s)$ is strictly proper
2. $\deg\big(a(s)\big) = \deg\big(b(a)\big)$ then $H(s)$ is proper
3. $\deg\big(a(s)\big) < \deg\big(b(a)\big)$ the $H(s)$ is inproper.

2.8 Frequency domain description

Beacuse of the similarity between the F-transform and the L-transform, the frequency domain description is similar to the operator domain description. Using the F-transform we get

$$Y(j\omega) = H(j\omega)U(j\omega)$$

where $Y(j\omega) = \mathcal{F}\{y(t)\}$ and $U(j\omega) = \mathcal{F}\{u(t)\}$.

2.9 State-space representation

The state space model consists of two set of equations, state equations and output equations. For CT LTI systems the general state space model is as follows:

$$\dot{x}(t) = Ax(t) + u(t) \qquad \text{(state equation)}$$

$$y(t) = Cx(t) + Du(t) \qquad \text{(output equation)}$$

with given initial condition $x(t_0) = t(0)$ and

$$x(t) \in \mathbb{R}^n \quad y(t) \in \mathbb{R}^p \quad u(t) \in \mathbb{R}^r$$

and

$$A \in \mathbb{R}^{n\times n} \quad B \in \mathbb{R}^{n\times r} \quad C \in \mathbb{R}^{p\times n} \quad D \in \mathbb{R}^{p\times r}.$$

The state space representation (SSR) consists of the constant matrices (A, B, C, D). The dimension of the SSR is $\dim x(t) = n$. The state space $\mathbb{X}$ is the set of all $x(t)$ states. Usually we assume that $D = 0$, since this can be achieved by re-scaling the input and output signals.

2.10 Transformation of states

Given two SSRs

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

and

$$\dot{\overline{x}}(t) = \overline{A}\overline{x}(t) + \overline{B}u(t)$$
$$y(t) = \overline{C}\overline{x}(t) + \overline{D}u(t)$$

related by the transformation

$$T \in \mathbb{R}^{n\times n} \quad \det T \neq 0$$

where $\overline{x} = Tx$, the relations between the matrices are

$$\overline{A} = TAT^{-1} \quad \overline{B} = TB \quad \overline{C} = CT^{-1} \quad \overline{D} = D.$$

Proof

Since T is invertible we get $\dot{x}(t) = T^{-1}\dot{\overline{x}}(t)$ and

$$T^{-1}\dot{\overline{x}}(t) = AT^{-1}\overline{x}(t) + Bu(t).$$

Then

$$\dot{\overline{x}}(t) = TAT^{-1}\overline{x}(t) + TBu(t)$$
$$y(t) = CT^{-1}\overline{x}(t) + Du.$$

We can see that indeed

$$\overline{A} = TAT^{-1} \quad \overline{B} = TB \quad \overline{C} = CT^{-1} \quad \overline{D} = D.$$

2.11 Computation of the transfer function from SSR

Given the SSR (A, B, C, D) the transfer function is

$$H(s) = C(sI - A)^{-1}B + D.$$

Proof

Taking the L-transform of the SSR we get (assuming $x(0) = 0$)

$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s).$$

From this we can compute

$$X(s) = (sI - A)^{-1}BU(s)$$
$$Y(s) = \big(C(sI - A)^{-1}B + D\big)U(s).$$

Since

$$Y(s) = H(s)U(s)$$

we can see that

$$H(s) = C(sI - A)^{-1}B + D.$$

2.12 Independence of transfer function

The transfer function is independent from the SSR, meaning the transfer function remains unchanged if we apply a transformation to the SSR.

Proof

We know that applying a T transformation the transformed matrices are

$$\overline{A} = TAT^{-1} \quad \overline{B} = TB \quad \overline{C} = CT^{-1} \quad \overline{D} = D.$$

Then for $\big(\overline{A}, \overline{B}, \overline{C}, \overline{D}\big)$ the transfer function is

$$H(s) = \overline{C}\big(sI - \overline{A}\big)^{-1}\overline{B} + \overline{D}.$$

Since

$$\big(sI - \overline{A}\big)^{-1} = \frac{1}{s}\left(I - \frac{\overline{A}}{s}\right)^{-1} = \frac{1}{s}\sum_{k=0}^{\infty}\frac{\overline{A}^k}{k!}$$

and

$$\overline{A}^k = \big(TAT^{-1}\big)^k = TA^kT^{-1}$$

we can see that

$$\big(sI - \overline{A}\big)^{-1} = T\frac{1}{s}\sum_{k=0}^{\infty}\frac{A^k}{k!}T^{-1} = T(sI - A)^{-1}T^{-1}.$$

Hence

$$H(s) = \overline{C}\big(sI - \overline{A}\big)^{-1}\overline{B} + \overline{D} = CT^{-1}T(sI - A)^{-1}T^{-1}TB + D = C(sI - A)^{-1}B + D.$$

2.13 Minimal SSR

A SSR is minimal if $\dim x$ is the smallest among all SSRs of a given system.

2.14 Solution of state equation

Given the state space model

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

we can compute the solution as

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau$$
$$y(t) = Cx(t) + Du(t).$$

Proof
We know that

$$X(s) = (sI - A)^{-1}BU(s)$$

and

$$(sI - A)^{-1} = \frac{1}{s}\sum_{k=0}^{\infty}\frac{A^k}{k!} \implies \mathcal{L}^{-1}\Big\{(sI - A)^{-1}\Big\} = \sum_{k=0}^{\infty}\frac{A^k}{k!}t^k = e^{At}.$$

Taking the inverse L-transform we get

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,\mathrm{d}\tau$$
$$y(t) = Cx(t) + Du(t).$$

2.15 Markov parameters

The Markov parameters of a CT system are

$$h_i = CA^{i-1}B \qquad i = 1, 2, \ldots$$

2.16 Independece of Markov parameters

The Markov parameters are independent from the SSR, meaning the Markov parameters remain unchanged if we apply a transformation to the SSR.

Proof
Since

$$\overline{A} = TAT^{-1} \quad \overline{B} = TB \quad \overline{C} = CT^{-1}$$

the parameters of the realization $(\overline{A}, \overline{B}, \overline{C})$ are

$$h_i = \overline{C}\overline{A}^{i-1}\overline{B} = CT^{-1}TA^{i-1}T^{-1}TB = CA^{i-1}B.$$

2.17 Theorem

Assuming $x(0) = 0$, $D = 0$ and $u(t) = \delta(t)$ we get

$$h(t) = Ce^{At}B = \sum_{k=0}^{\infty} \frac{CA^kB}{k!}t^k = \sum_{k=0}^{\infty} \frac{h_{k+1}}{k!}t^k.$$

Proof

Since

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}B\delta(\tau)\,\mathrm{d}\tau$$
$$h(t) = Cx(t) + D\delta(t)$$

with $x(0) = 0$ and $D = 0$ we get

$$h(t) = C\int_0^t e^{A(t-\tau)}B\delta(\tau)\,\mathrm{d}\tau = Ce^{At}B = \sum_{k=0}^{\infty} \frac{CA^kB}{k!}t^k = \sum_{k=0}^{\infty} \frac{h_{k+1}}{k!}t^k.$$

2.18 State observability (problem statement)

Given (A, B, C), u and y, we want to determine x.

2.19 Observability matrix

The observality matrix with given SSR (A, B, C) is

$$\mathcal{O}_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

2.20 Theorem

Given (A, B, C) is observable if and only if $\operatorname{rank} \mathcal{O}_n = n$. (Note that this is a realization property.)

Proof

From the SSR output equation we get

$$y = Cx$$
$$\dot{y} = C\dot{x} = CAx + CBu$$
$$\ddot{y} = C\ddot{x} = CA(Ax + Bu) + CB\dot{u} = CA^2x + CABu + CB\dot{u}$$
$$\vdots$$
$$y^{(n-1)} = Cx^{(n-1)} = CA^{n-1}x + \sum_{k=1}^{n-1} CA^{n-1-k}Bu^{(k)}.$$

This can be rewritten as

$$\begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x + \begin{bmatrix} 0 & 0 & \dots & 0 \\ CB & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{n-2}B & CA^{n-3}B & \dots & 0 \end{bmatrix} \begin{bmatrix} u \\ \dot{u} \\ \vdots \\ u^{(n-1)} \end{bmatrix}$$

i.e.

$$\dot{Y}(t) = \mathcal{O}_n x(t) + T\dot{U}(t).$$

We can assume that for the initial state $x(0)$ the derivatives of the input are zero, i.e. $\dot{U}(0) = 0$. In case of nonzero initial conditions we can arrange the equation in another form to get the zero initial condition. With $\dot{U}(0) = 0$ we get

$$\dot{Y}(0) = \mathcal{O}_n x(0).$$

For the above equation to be solvable we need $\mathcal{O}_n$ to be invertable, i.e. $\det \mathcal{O}_n \neq 0$ which means that $\operatorname{rank} \mathcal{O}_n = n$.

2.21 State controllability (problem statement)

Given a SSR of the system, we want to drive the state $x(t_1)$ to $x(t_2)$ with appropriate input in finite time.

2.22 Controllability matrix

The observality matrix with given SSR (A, B, C) is

$$\mathcal{C}_n = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}.$$

2.23 Theorem

Given (A, B, C) is controllable if and only if $\operatorname{rank} \mathcal{C}_n = n$. (Note that this is a realization property.)

Proof

Applying the Dirac-delta function as input we get $h(t) = e^{At}B$ (with $C = I$). We can see that with an input $\delta^{(k)}(t)$ the system response will be $h^{(k)}(t) = A^k h(t)$. Now applying an input of the form

$$u(t) = \sum_{k=1}^{n} g_k \delta^{(k-1)}(t)$$

then

$$x(0_+) = x(0) + \sum_{k=1}^{n} g_k h^{(k-1)}(0) = x(0) + \sum_{k=1}^{n} g_k A^{k-1} B.$$

Assuming $x(0) = 0$ we get

$$x(0_+) = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} = \mathcal{C}_n G.$$

With arbitrary $x(0_+)$ for this equation to be solvable we need $\mathcal{C}_n$ to be invertable, i.e. $\det \mathcal{C}_n \neq 0$ which means that $\operatorname{rank} \mathcal{C}_n = n$.

2.24 Kalman rank condition

If $\dim \mathbb{X} = n$ then $\operatorname{rank} \mathcal{O}_n = \operatorname{rank} \mathcal{C}_n = n$.

2.25 Diagonal form realization of SSR

A diagonal form realization (DSSR) is a realization (A, B, C) of the following form:

$$\dot{x}(t) = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} x(t) + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix} u(t).$$

2.26 Observability in DSSR

The realization is observable if and only if $\forall \lambda_i \neq \lambda_j$ and $\forall c_k \neq 0$.

Proof
The observability matrix

$$\mathcal{O}_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & \dots & c_n \\ \lambda_1 c_1 & \lambda_2 c_2 & \dots & \lambda_n c_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} c_1 & \lambda_2^{n-1} c_2 & \dots & \lambda_n^{n-1} c_n \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} c_1 & 0 & \dots & 0 \\ 0 & c_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_n \end{bmatrix}.$$

Since the first matrix is a Vandermonde matrix (or at least its determinant can be written as a Vandermonde determinant)

$$\det \mathcal{O}_n = \prod_{i<j} (\lambda_i - \lambda_j) \prod_{k=1}^{n} c_k.$$

We need $\det \mathcal{O}_n \neq 0$ for the observability. That means if $\forall \lambda_i \neq \lambda_j$ and $\forall c_k \neq 0$ then the realization is observable.

2.27 Controllability in DSSR

The realization is controllable if and only if $\forall \lambda_i \neq \lambda_j$ and $\forall b_k \neq 0$.

Proof
The controllability matrix

$$\mathcal{C}_n = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = \begin{bmatrix} b_1 & \lambda_1 b_1 & \dots & \lambda_1^{n-1} b_1 \\ b_2 & \lambda_2 b_2 & \dots & \lambda_2^{n-1} b_2 \\ \vdots & \vdots & \ddots & \vdots \\ b_n & \lambda_n b_n & \dots & \lambda_n^{n-1} b_n \end{bmatrix} =$$

$$= \begin{bmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_n \end{bmatrix} \begin{bmatrix} 1 & \lambda_1 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \dots & \lambda_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \dots & \lambda_n^{n-1} \end{bmatrix}.$$

Since the second matrix is a Vandermonde matrix

$$\det \mathcal{C}_n = \prod_{k=1}^{n} b_k \prod_{i<j} (\lambda_i - \lambda_j).$$

We need $\det \mathcal{C}_n \neq 0$ for the controllability. That means if $\forall \lambda_i \neq \lambda_j$ and $\forall b_k \neq 0$ then the realization is controllable.

2.28 Transfer function of DSSR

$$H(s) = C(sI - A)^{-1} B = \sum_{i=1}^{n} \frac{c_i b_i}{s - \lambda_i} = \frac{b(s)}{a(s)}.$$

Proof

$$H(s) = C(sI-A)^{-1}B = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix} \begin{bmatrix} s-\lambda_1 & 0 & \dots & 0 \\ 0 & s-\lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s-\lambda_n \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} =$$

$$= \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix} \begin{bmatrix} \frac{1}{s-\lambda_1} & 0 & \dots & 0 \\ 0 & \frac{1}{s-\lambda_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{s-\lambda_n} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \sum_{i=1}^{n} \frac{c_i b_i}{s-\lambda_i}$$

2.29 Hankel-matrix

A Hankel-matrix is a block matrix of the following form

$$H(1,n-1) = \begin{bmatrix} CB & CAB & \dots & CA^{n-1}B \\ CAB & CA^2B & \dots & CA^nB \\ \vdots & \vdots & \ddots & \vdots \\ CA^{n-1}B & CA^nB & \dots & CA^{2n-2}B \end{bmatrix}.$$

Note that the Hankel-matrix consists of Markov parameters of the system.

2.30 Independence of the Hankel matrices

The Hankel matrices are independent from the SSR, meaning the Hankel matrices remain unchanged if we apply a transformation to the SSR.

Proof

Since we already proved the independence of the Markov parameters, we get the desired result immediately.

2.31 Conditions for joint controllability and observability of SISO LTI systems

2.31.1 Realization independence of joint controllability and observability

If a given system's n-th order realization (A, B, C) is controllable and observable then all other n-th order realizations are controllable and observable i.e. joint controllability and observability is a system property.

Proof

Notice that

$$H(1,n-1) = \mathcal{O}_n \mathcal{C}_n.$$

If the realization is jointly controllable and observable then $\mathcal{O}_n$ and $\mathcal{C}_n$ are nonsingular which means that $H(1,n-1)$ is nonsingular. Since we proved that the Hankel matrix is independent from the realization we get that $H(1,n-1)$ is nonsingular for any realization. This implies that in all realization $\mathcal{O}_n$ and $\mathcal{C}_n$ are nonsingular which means that all realizations are jointly controllable and observable.

2.31.2 Observer form realization of SSR

The controller form realization of a SISO LTI system (A_c, B_c, C_c) is as follows:

$$\dot{x}(t) = \begin{bmatrix} -a_{n-1} & 1 & \dots & 0 & 0 \\ -a_{n-2} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_1 & 0 & \dots & 1 & 0 \\ -a_0 & 0 & \dots & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \\ b_0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix} x(t)$$
$$H(s) = \frac{\sum_{k=0}^{n-1} b_k s^k}{\sum_{k=0}^{n-1} a_k s^k + s^n}.$$

2.31.3 Theorem

The observer form realization is always observable.

Proof
Notice that the observability matrix will be a lower trianular matrix with ones in the diagonal. That means that $\det \mathcal{O}_n \neq 0$ hence the realization is observable.

2.31.4 Controller form realization of a SSR

The controller form realization of a SISO LTI system (A_c, B_c, C_c) is as follows:

$$\dot{x}(t) = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \dots & -a_1 & -a_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} b_{n-1} & b_{n-2} & \dots & b_1 & b_0 \end{bmatrix} x(t)$$
$$H(s) = \frac{\sum_{k=0}^{n-1} b_k s^k}{\sum_{k=0}^{n-1} a_k s^k + s^n}.$$

2.31.5 Theorem

The controller form realization is always controllable.

Proof
Notice that the controllability matrix will be an upper trianular matrix with ones in the diagonal. That means that $\det \mathcal{C}_n \neq 0$ hence the realization is controllable.

2.31.6 Theorem

If the controller form realization is jointly controllable and observable if and only if $a(s)$ and $b(s)$ are realtive primes ($H(s)$ is irreducible).

Proof
Suppose that the polynomial $b(s)$ is as follows

$$b(s) = \sum_{k=1}^{n} b_k s^k.$$

Let e_i be the i-th uniy vector. Let's observe the result of the

$$\tilde{I}_n b(A_c) = \begin{bmatrix} e_n^{\mathrm{T}} \\ e_{n-1}^{\mathrm{T}} \\ \vdots \\ e_1^{\mathrm{T}} \end{bmatrix} b(A_c)$$

product. It is easy to see that

$$e_i^{\mathrm{T}} A_c = \begin{cases} \begin{bmatrix} -a_{n-1} & -a_{n-2} & \dots & -a_0 \end{bmatrix} & \text{if } i = 1 \\ e_{i-1}^{\mathrm{T}} & \text{if } i \geq 2. \end{cases}$$

From this we get

$$e_n^{\mathrm{T}} b(A_c) = \sum_{k=0}^{n-1} e_n^{\mathrm{T}} b_k A_c^k = \sum_{k=0}^{n-1} e_{k+1}^{\mathrm{T}} b_k = \begin{bmatrix} b_{n-1} & b_{n-2} & \dots & b_0 \end{bmatrix} = C_c$$

$$e_{n-1}^{\mathrm{T}} b(A_c) = e_n^{\mathrm{T}} A_c b(A_c) = e_n^{\mathrm{T}} b(A_c) A_c = C_c A_c$$

and in general

$$e_i^{\mathrm{T}} b(A_c) = C_c A_c^{n-i}.$$

Using this we get

$$\tilde{I}_n b(A_c) = \begin{bmatrix} C_c \\ C_c A_c \\ \vdots \\ C_c A_c^{n-1} \end{bmatrix} = \mathcal{O}_n.$$

For the observability we need $\mathcal{O}_n \neq 0$. We know that

$$\det \mathcal{O}_n = \det b(A_c) = \prod_{i=1}^{n} b(\lambda_i)$$

where λ_i-s are the eigenvalues of A_c i.e. the roots of $a(s) = \det(sI - A_c)$. We get that $a(s)$ and $b(s)$ are relative primes if and only if the observability matrix is of full rank. This means that the controller form realization is jointly controllable and observable if and only if $a(s)$ and $b(s)$ are relative primes i.e. $H(s)$ is irreducible.

2.31.7 Theorem

$H(s)$ is irreducible if and only if all n-th order realizations are jointly controllable and observable.

Proof

Let's suppose that $H(s)$ is irreducible. This implies that the controller form is jointly controllable and observable from which we get that all realizations are jointly controllable and observable since it is a system property.
Now assuming that all realizations are jointly controllable and observable we get that the contoller form is also jointly controllable and observable. This implies that $H(s)$ is irreducible.

2.31.8 Minimal realization

A realization (A, B, C) of dimension n is minimal if all other realization has a dimension higher than n.

2.31.9 Theorem

$H(s)$ is irreducible if and only if $\forall(A, B, C)$ realizations are minimal.

Proof

Let us suppose that $H(s)$ is irreducible but the realization is not minimal. Then we can find another realization $\left(\overline{A}, \overline{B}, \overline{C}\right)$ which is minimal. Then the degree of the denominator of $\overline{H}(s)$ is less then the degree of denominator of $H(s)$ which is a contradiction.
Suppose we have a minimal realization but $H(s)$ is reducible. From the simplified transfer function we can obtain anouther realization with smaller order then the minimal realization. This is a contradiction.

2.31.10 Theorem

A realization (A, B, C) is minimal if and only if the system is jointly controllable and observable.

Proof

Let us suppose that (A, B, C) is a minimal realization. Then the transfer function $H(s)$ is irreducible which means that the system is jointly controllable and observable.
Suppode that the system is jointy controllable and observable. This implies that the transfer function is irreducibl. From this we get that any realization is minimal i.e. (A, B, C) is minimal.

2.31.11 Theorem

Any two minimal realizations can be connected by a unique, invertible similarity transformation.

Proof
Let the two realization be (A_1, B_1, C_1) and (A_2, B_2, C_2). If the transformation T exists we know that

$$A_2 = TA_1T^{-1} \qquad B_2 = TB_1 \qquad C_2 = C_1T^{-1}.$$

Examine the observability matrices of the realizations!

$$\mathcal{O}_2 = \begin{bmatrix} C_2 \\ C_2A_2 \\ \vdots \\ C_2A_2^{n-1} \end{bmatrix} = \begin{bmatrix} C_1T^{-1} \\ C_1A_1T^{-1} \\ \vdots \\ C_1A_1^{n-1}T^{-1} \end{bmatrix} = \mathcal{O}_1T^{-1}$$

From this we get $T = \mathcal{O}_2^{-1}\mathcal{O}_1$ is an adequate transformation matrix since minimal realizations are observable i.e. T exists and is invertible.
Similarly the controllability matrices

$$\mathcal{C}_2 = \begin{bmatrix} B_2 & A_2B_2 & \dots A_2^{n-1}B_2 \end{bmatrix} = \begin{bmatrix} TB_1 & TA_1B_1 & \dots TA_1^{n-1}B_1 \end{bmatrix} = T\mathcal{C}_1.$$

From this we get $T = \mathcal{C}_2\mathcal{C}_1^{-1}$ is an adequate transformation matrix since minimal realizations are controllable i.e. T exists and is invertible.
We constructed the

$$T = \mathcal{O}_2^{-1}\mathcal{O}_1 = \mathcal{C}_2\mathcal{C}_1^{-1}$$

transformation matrix which connects the two minimal realizations.

2.32 General decomposition theorem

For any given (A, B, C) realization the exists an invertible similarity transformation which moves the system to the realization $(\overline{A}, \overline{B}, \overline{C})$ where

$$\overline{x} = \begin{bmatrix} \overline{x}_{co} & \overline{x}_{c\overline{o}} & \overline{x}_{\overline{c}o} & \overline{x}_{\overline{co}} \end{bmatrix}^T$$

$$\overline{A} = \begin{bmatrix} \overline{A}_{co} & 0 & \overline{A}_{13} & 0 \\ \overline{A}_{21} & \overline{A}_{c\overline{o}} & \overline{A}_{23} & \overline{A}_{24} \\ 0 & 0 & \overline{A}_{\overline{c}o} & 0 \\ 0 & 0 & \overline{A}_{43} & \overline{A}_{\overline{co}} \end{bmatrix} \qquad \overline{B} = \begin{bmatrix} \overline{B}_{co} \\ \overline{B}_{c\overline{o}} \\ 0 \\ 0 \end{bmatrix}$$

$$\overline{C} = \begin{bmatrix} \overline{C}_{co} & 0 & \overline{C}_{\overline{c}o} & 0 \end{bmatrix}.$$

2.32.1 Controllable and observable subsystem

The realization of the controllable and observable subsystem is $(\overline{A}_{co}, \overline{B}_{co}, \overline{C}_{co})$. This realization is minimal.

2.32.2 Observable subsystem

The realization is

$$\left(\begin{bmatrix} \overline{A}_{co} & \overline{A}_{13} \\ 0 & \overline{A}_{\overline{c}o} \end{bmatrix} \begin{bmatrix} \overline{B}_{co} \\ 0 \end{bmatrix} \begin{bmatrix} \overline{C}_{co} & \overline{C}_{\overline{c}o} \end{bmatrix} \right).$$

2.32.3 Controllable subsystem

The realization is

$$\left(\begin{bmatrix} \overline{A}_{co} & 0 \\ \overline{A}_{21} & \overline{A}_{c\overline{o}} \end{bmatrix} \begin{bmatrix} \overline{B}_{co} \\ \overline{B}_{c\overline{o}} \end{bmatrix} \begin{bmatrix} \overline{C}_{co} & \overline{C}_{c\overline{o}} \end{bmatrix} \right).$$

2.32.4 Uncontrollable and unobservable subsystem

The realization is

$$\left([\overline{A}_{\overline{co}}] \quad [0] \quad [0] \right).$$

2.33 External or BIBO stability of SISO LTI systems

A system is externally or BIBO stable if

$$\|u(t)\| \leq M_1 < \infty \implies \|y(t)\| \leq M_2 < \infty$$

where $\|.\|$ is a signal norm.

2.34 Theorem

A system is externally or BIBO stable if and only if

$$\int_0^\infty |h(t)| \,\mathrm{d}t \leq M < \infty.$$

Proof
Let us suppose that

$$\int_0^\infty |h(t)| \,\mathrm{d}t \leq M < \infty$$

hold. Then

$$|y(t)| = \left| \int_0^\infty h(\tau)u(t-\tau)\,\mathrm{d}\tau \right| \leq \int_0^\infty |h(\tau)||u(t-\tau)|\,\mathrm{d}\tau \leq M_1 M = M_2.$$

We got that $y(t)$ is bounded indeed.
Now suppose that $\int_0^\infty |h(\tau)|\,\mathrm{d}\tau = \infty$ but the output is bounded for any bounded input. Let us select the input as

$$u(t-\tau) = \operatorname{sgn} h(\tau).$$

Then

$$y(t) = \int_0^\infty h(\tau)u(t-\tau)\,\mathrm{d}\tau = \int_0^\infty |h(\tau)|\,\mathrm{d}\tau = \infty$$

which is a contradiction.

2.35 Internal or asymptotic stability of LTI systems

A realization (A, B, C) is internally or asymptotically stable if the solution $x(t)$ of the equation

$$\dot{x}(t) = Ax(t)$$

where $x(t_0) \neq 0$ fulfills

$$\lim_{t\to\infty} x(t) = 0.$$

2.36 Stability matrix

$A \in \mathbb{R}^{n\times n}$ is a stability matrix if all of its eigenvalues have strictly negative real parts.

2.37 Theorem

A CT LTI system is internally or asymptotically stable if and only if A is a stability matrix.

Proof
Apply the invertible T transformation that transform A to a diagonal form. We know that

$$\overline{A} = TAT^{-1} = \langle \lambda_k \rangle$$

where λ_k are the eigenvalues of A. We know that the solution of $\dot{\overline{x}}(t) = \overline{A}\overline{x}(t)$ is

$$\overline{x}(t) = e^{\overline{A}t}\overline{x}(0).$$

From this we get

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} T^{-1} e^{\overline{A}t} T^{-1} x(0) = \lim_{t\to\infty} T^{-1} \langle e^{\lambda_k t} \rangle T^{-1} x(0) =$$
$$= \lim_{t\to\infty} T^{-1} \Big\langle e^{a_k t} \big(\cos(b_k t) + j \sin(b_k t) \big) \Big\rangle T^{-1} x(0).$$

It is easy to see that

$$\lim_{t\to\infty} x(t) = 0$$

holds if and only if $\forall \operatorname{Re} \lambda_k < 0$.

2.38 Theorem

Internal or asymptotic stability is a system property.

Proof
Since the eigenvalues of A are invariant under transformation we immediately get the desired result.

2.39 Theorem

Internal or asymptotic stability implies external or BIBO stability.

Proof
The solutions of the state equations for a realization (A, B, C) is

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B u(\tau)\, \mathrm{d}\tau$$
$$y(t) = Cx(t).$$

If we feed the system a bounded input $\big|u(t)\big| \le M$ then

$$x(t) \le e^{At} x(0) + M \int_0^t e^{A(t-\tau)} B\, \mathrm{d}\tau = e^{At} \left(x(0) + M \int_0^t e^{-A\tau} B\, \mathrm{d}\tau \right) =$$
$$= e^{At} \left(x(0) - MA^{-1} e^{-A\tau} B \Bigg|_0^t \right) = e^{At} \Big(x(0) - MA^{-1} e^{-At} B + MA^{-1} B \Big) =$$
$$= e^{At} \Big(x(0) + MA^{-1} B \Big) - MA^{-1} B.$$

Since the system is internal stable $\lim_{t\to\infty} e^{At} = 0$ i.e. $x(t) \le -MA^{-1}B$. This means that $y(t)$ are bounded i.e. the system is BIBO stable.

2.40 Stability of autonomous nonlinear systems

Consider the autonomous nonlinear system

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n \quad f : \mathbb{R}^n \mapsto \mathbb{R}^n$$

with an equilibrium point $f(x^*) = 0$.

1. x^* is a stable equilibrium point if for $\forall\varepsilon > 0$ there exists $\delta > 0$ ($\delta < \varepsilon$) such that $\|x^* - x(0)\| < \delta$ implies $\|x^* - x(t)\| < \varepsilon$.
2. x^* is an asymptotically stable equilibrium point if x^* is stable and $\lim_{t\to\infty} x(t) = x^*$.
3. x^* is an unstable equilibrium point if it is not stable.
4. x^* is a locally (asymptotically) stable equilibrium point if there exists a neighborhood U of x^* within which the (asymptotic) stability conditions hold.
5. x^* is globally (asymptotically) stable if $U = \mathbb{R}^n$.

2.41 Lyapunov function

The generelized energy or Lyapunov function is $V(x)$ with the following properties:

1. $V : \mathbb{R}^n \mapsto \mathbb{R}$
2. $V(x) > 0$ if x is not an equilibrium point. For x^* equilibrium point $V(x^*) = 0$ holds.
3. $\dot{V}(x) = \frac{\mathrm{d}}{\mathrm{d}t}V(x) = \frac{\partial V}{\partial x}\dot{x}(t) = \frac{\partial V}{\partial x}f(x) \leq 0$

2.42 Lyapunov theorem

If there exists a Lyapunov function then

1. x^* is a stable equilibrium point if $f(x^*) = 0$ and $V(x^*)$ holds
2. x^* is an asymptotically stable equilibrium point if $\dot{V}(x) < 0$
3. x^* is a locally (asymptotically) stable equilibrium point if the conditions only hold in a neighborhood U of x^*.

2.43 Lyapunov criterion for LTI systems

A is a stability matrix if and only if for any given positive deifinite symmetric matrix Q there exists a positive definite symmetric matrix P such that

$$A^{\mathrm{T}}P + PA = -Q.$$

Proof

Let us suppose that for $\forall Q > 0$ there exists $P > 0$ such that $A^{\mathrm{T}}P + PA = -Q$. Then for $V(x) = x^{\mathrm{T}}Px$

$$\dot{V}(x) = \dot{x}^{\mathrm{T}}Px + x^{\mathrm{T}}P\dot{x} = x^{\mathrm{T}}A^{\mathrm{T}}Px + x^{\mathrm{T}}PAx = x^{\mathrm{T}}\Big(A^{\mathrm{T}}P + PA\Big)x = -x^{\mathrm{T}}Qx < 0.$$

From the Lyapunov theorem we immediately get that the system is asymptotically stable i.e. A is a stability matrix.

Now suppose that A is a stability matrix. Then for

$$P = \int_0^\infty e^{A^{\mathrm{T}}t}Qe^{At}\,\mathrm{d}t$$

$$A^{\mathrm{T}}P + PA = \int_0^\infty A^{\mathrm{T}}e^{A^{\mathrm{T}}t}Qe^{At}\,\mathrm{d}t + \int_0^\infty e^{A^{\mathrm{T}}t}Qe^{At}A\,\mathrm{d}t =$$

$$= e^{A^{\mathrm{T}}t}Qe^{At}\Big|_0^\infty - \int_0^\infty e^{A^{\mathrm{T}}t}QAe^{At}\,\mathrm{d}t + \int_0^\infty e^{A^{\mathrm{T}}t}QAe^{At}\,\mathrm{d}t = e^{A^{\mathrm{T}}t}Qe^{At}\Big|_0^\infty = -Q.$$

2.44 Interconnection of subsystems

(Block diagrams)

2.44.1 Serial interconnection

$$H(s) = H_1(s)H_2(s)$$
$$h(t) = (h_1 * h_2)(t)$$

2.44.2 Parallel interconnetion

$$H(s) = H_1(s) + H_2(s)$$
$$h(t) = h_1(t) + h_2(t)$$

2.44.3 General negative feedback

$$H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

2.45 Important terms

1. Proportional term
   A proportional term has a transfer function $H(s) = K_p$ where K_p is the proportional gain.
2. Integral term
   An integral term has a transfer function $H(s) = \frac{1}{T_i s}$ where T_i is the integral time. For the input $u(t)$ the output is $y(t) = \frac{1}{T_i} \int_0^t u(\tau)\,\mathrm{d}\tau$.
3. Derivative term
   A derivative term has a transfer function $H(s) = T_d s$ where T_d is the derivative gain. For the input $u(t)$ the output is $y(t) = T_d \dot{u}(t)$. (Note that a derivative term cannot be used in a fully causal system.)

2.46 PID controller

As any other controller the PID controllers aim is to eliminate the error between the input and the output. We use a general negative feedback to feed the error to the controller which is in a serial interconnection with the system itself.
(Block diagram)

2.47 Ziegler-Nichols method

The Ziegler-Nichols method is a tuning method for PID controllers in order to determine the optimal proportional, integral and derivative gains. First we increase the proportional gain until it reaches a ultimate gain K_u with an oscillation period T_u.

Control type	K_p	T_i	T_d
P	$\frac{1}{2}K_u$	-	-
PI	$\frac{9}{20}K_u$	$\frac{5}{6}T_u$	-
PD	$\frac{4}{5}K_u$	-	$\frac{1}{8}T_u$
PID (fast)	$\frac{3}{10}K_u$	$\frac{1}{2}T_u$	$\frac{1}{8}T_u$
PID (small overshoot)	$\frac{1}{3}K_u$	$\frac{1}{2}T_u$	$\frac{1}{3}T_u$
PID (without overshoot)	$\frac{1}{5}K_u$	$\frac{3}{10}T_u$	$\frac{1}{2}T_u$

2.48 General form of the static linear full state variable feedback

Let us have an SSR (A, B, C) of a LTI system in the form

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t).$$

Modifying the system by a static linear full state feedback

$$v = u + kx$$

where $k = \begin{bmatrix} k_1 & k_2 & \dots k_n \end{bmatrix}$. The state space representation of the closed loop system is

$$\dot{x}(t) = (A - Bk)x(t) + Bv(t)$$

$$y(t) = Cx(t)$$

with a closed loop polynomial

$$\alpha(s) = \det(sI - A + Bk).$$

2.49 Bass-Gura formula

$$\det\left\{\begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}\right\} = \det M_1 \det\bigl(M_4 - M_3 M_1^{-1} M_2\bigr) = \det M_4 \det\bigl(M_1 - M_2 M_4^{-1} M_3\bigr)$$

2.50 Pole placement feedback

State feedback can arbitrarily relocate the poles of the system with realization (A, B, C) if and only if the SSR is controllable using a state feedback parameter

$$k = (\alpha - a) T_l^{-\mathrm{T}} \mathcal{C}^{-1}$$

where T_l is a Toeplitz matrix of the form

$$T_l = \begin{bmatrix} 1 & 0 & \dots & 0 \\ a_1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \dots & 1 \end{bmatrix}$$

$a(s)$ is the open loop polynomial and $\alpha(s)$ is the desired closed loop polynomial.

Proof

Using the Bass-Gura formula we get

$$\det\left\{\begin{bmatrix} sI - A & B \\ -k & 1 \end{bmatrix}\right\} = \det(sI - A) \det\left(1 + k(sI - A)^{-1} B\right) = \det(sI - A + Bk)$$

which can be rewritten as

$$a(s)(1 + k(sI - A)^{-1} B) = \det(sI - A + Bk) = \alpha(s).$$

From this we get

$$\alpha(s) - a(s) = a(s) k (sI - A)^{-1} B.$$

Using

$$(sI - A)^{-1} = \frac{1}{a(s)} \sum_{i=1}^{n} s^{n-i} \sum_{j=0}^{i-1} a_{i-1-j} A^j$$

we get

$$\alpha(s) - a(s) = k\sum_{i=1}^{n} s^{n-i}\sum_{j=0}^{i-1} a_{i-1-j}A^j B = k\begin{bmatrix}B & AB & \dots & A^{n-1}B\end{bmatrix}\begin{bmatrix}1 & a_1 & \dots & a_{n-1}\\ 0 & 1 & \dots & a_{n-2}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 1\end{bmatrix} = k\mathcal{C}T_l^{\mathrm{T}}.$$

Since $\det T_k^{\mathrm{T}} = 1$ the nonsingularity of the right hand side only depends on the controllability matrix i.e. k is computable as

$$k = (\alpha - a)T_l^{\mathrm{T}}\mathcal{C}^{-1}$$

if and only if $\mathcal{C}$ is nonsingular.

2.51 Pole placement design in controller form

Given a controller form realization (A_c, B_c, C_c) the feedback parameter can be calculated as

$$k_c = \alpha - a.$$

Proof

Since

$$\alpha(s) = \det(sI - A_c + B_c k) = \sum_{i=0}^{n}(a_i + k_{ci})s^{n-i}$$

we immediately get the desired result.

2.52 Linear quadratic regulator (problem statement)

Given a CT LTI MIMO system with the SSR (A, B, C)

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

we define a functional

$$\mathcal{J}(x,u) = \frac{1}{2}\int_0^T \Big(x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)Ru(t)\Big)\,\mathrm{d}t$$

where Q is the positive semidefinite state weighting matrix and R is the positive semidefinite control weighting matrix (i.e. the functional measures a wheighted norm and the control energy). We want to find the optimal $u(t)$ that minimizes $J(x, u)$.

2.52.1 Solution of the LQR problem

The optimal input that minimizes

$$\mathcal{J}(x,u) = \frac{1}{2}\int_0^T \Big(x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)Ru(t)\Big)\,\mathrm{d}t$$

is

$$u(t) = -R^{-1}B^{\mathrm{T}}K(t)x(t)$$

where $K(t)$ is the solution of the Matrix Riccati Differencial Equation of the form

$$\dot{K} + KA + A^{\mathrm{T}}K - KBR^{-1}B^{\mathrm{T}}K + Q = 0.$$

Proof

Generalizing the problem we want to minimize the objective function

$$\mathcal{J}(x,u) = \int_0^T F(x,u,t)\,\mathrm{d}t$$

with respect to u with the contraint $f(x,u,t) - \dot{x} = 0$. Using a Langrange multiplier we get

$$\mathcal{J}(x,\dot{x},u,t) = \int_0^T \Big(F(x,u,t) + \lambda^{\mathrm{T}}(t)\big(f(x,u,t) - \dot{x}\big)\Big)\,\mathrm{d}t\,.$$

Further define the Hamiltonian

$$H(x,u,t) = F(x,u,t) + \lambda^{\mathrm{T}}(t) f(x,u,t).$$

Integrating by parts

$$\lambda^{\mathrm{T}} x\Big|_0^T = \int_0^T \dot{\lambda}^{\mathrm{T}} x\,\mathrm{d}t + \int_0^T \lambda^{\mathrm{T}} \dot{x}\,\mathrm{d}t$$

and substituting H we obtain

$$\mathcal{J}(x,\dot{x},u,t) = \int_0^T \Big(H(x,u,t) - \lambda^{\mathrm{T}}(t)\dot{x}(t)\Big)\,\mathrm{d}t = \int_0^T \big(H + \dot{\lambda}^{\mathrm{T}} x\big)\,\mathrm{d}t - \lambda^{\mathrm{T}} x\Big|_0^T.$$

Let the variations of $x(t)$ and $u(t)$ be

$$x(\alpha,t) = x(t) + \alpha\eta(t)$$
$$u(\beta,t) = u(t) + \beta\gamma(t)$$

where $\eta(0) = \eta(T) = \gamma(0) = \gamma(T) = 0$ granting that $x(\alpha,0) = x(0)$, $x(\alpha,T) = x(T)$, $u(\beta,0) = u(0)$ and $u(\beta,T) = u(T)$. The variation of the objective function is

$$\mathcal{I}(\alpha,\beta) = \mathcal{J}\big(x(\alpha,t),u(\beta,t),t\big) = \int_0^T \big(H\big(x(\alpha,t),u(\beta,t),t\big) + \dot{\lambda}^{\mathrm{T}}(t)x(\alpha,t)\big)\,\mathrm{d}t - \lambda^{\mathrm{T}}(t)x(\alpha,t)\Big|_0^T.$$

For u to be minimizing we need $\nabla\mathcal{I} = 0$.

$$\frac{\partial\mathcal{I}}{\partial\alpha} = -\lambda^{\mathrm{T}}\frac{\partial x}{\partial\alpha}\Big|_0^T + \int_0^T \left(\frac{\partial H}{\partial x}\frac{\partial x}{\partial\alpha} + \dot{\lambda}^{\mathrm{T}}\frac{\partial x}{\partial\alpha}\right)\mathrm{d}t = \int_0^T \left(\frac{\partial H}{\partial x} + \dot{\lambda}^{\mathrm{T}}\right)\eta\,\mathrm{d}t = 0$$

which holds if and only if $\frac{\partial H}{\partial x} + \dot{\lambda}^{\mathrm{T}} = 0$.

$$\frac{\partial\mathcal{I}}{\partial\beta} = \int_0^T \frac{\partial H}{\partial u}\frac{\partial u}{\partial\beta}\,\mathrm{d}t = \int_0^T \frac{\partial H}{\partial u}\gamma\,\mathrm{d}t = 0$$

which holds if and only if $\frac{\partial H}{\partial u} = 0$. These are the so called Euler-Lagrange equations

$$\frac{\partial H}{\partial x} + \dot{\lambda}^{\mathrm{T}} = 0$$
$$\frac{\partial H}{\partial u} = 0.$$

Proceeding to the original problem we have

$$f(x,u,t) = Ax + Bu$$
$$F(x,u,t) = \frac{1}{2}\Big(x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru\Big)$$
$$H(x,u,t) = \frac{1}{2}\Big(x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru\Big) + \lambda^{\mathrm{T}}(Ax + Bu).$$

The Euler-Lagrange equations are in the form using $\frac{\partial}{\partial x}x^{\mathrm{T}}Qx = 2x^{\mathrm{T}}Q$

$$\dot{\lambda}^{\mathrm{T}} + x^{\mathrm{T}}Q + \lambda^{\mathrm{T}}A = 0$$

$$u^{\mathrm{T}}R + \lambda^{\mathrm{T}}B = 0.$$

This can be rewritten as

$$\dot{\lambda} = -Qx - A^{\mathrm{T}}\lambda$$
$$u = -R^{-1}B^{\mathrm{T}}\lambda.$$

Using the original system model

$$\dot{x} = Ax + Bu$$

we get the Hammerstein costate differential equation

$$\begin{bmatrix}\dot{x}(t)\\ \dot{\lambda}(t)\end{bmatrix} = \begin{bmatrix}A & -BR^{-1}B^{\mathrm{T}}\\ -Q & -A^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}x(t)\\ \lambda(t)\end{bmatrix}.$$

Using a lemma that states if (A, B) is controllable and (C, A) is observable then $\lambda(t) = K(t)x(t)$ where $K(t) \in \mathbb{R}^{n\times n}$ we get

$$\dot{\lambda} = -Qx - A^{\mathrm{T}}\lambda = \dot{K}x + K\Big(Ax - BR^{-1}B^{\mathrm{T}}\lambda\Big).$$

This can be rewritten as

$$\Big(\dot{K} + KA + A^{\mathrm{T}}K - KBR^{-1}B^{\mathrm{T}}K + Q\Big)x = 0.$$

From this we get the Matrix Riccati Differential Equation of the form

$$\dot{K} + KA + A^{\mathrm{T}}K - KBR^{-1}B^{\mathrm{T}}K + Q = 0.$$

2.52.2 Stationary solution

In the special case when $T \to \infty$ the functional is

$$\mathcal{J}(x, u) = \frac{1}{2}\int_0^{\infty}\Big(x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)Ru(t)\Big)\,\mathrm{d}t$$

which can be minimized with an input

$$u(t) = -R^{-1}B^{\mathrm{T}}Kx(t)$$

where constant matrix K is the solution of the Control Algebraic Riccati Equation (CARE) of the form

$$KA + A^{\mathrm{T}}K - KBR^{-1}B^{\mathrm{T}}K + Q = 0.$$

Proof

Using a lemma that states if $t \to \infty$ then $K(t) = K$ i.e. $\dot{K} = 0$ the Matrix Riccati Differential Equation can be written as a CARE of the form

$$KA + A^{\mathrm{T}}K - KBR^{-1}B^{\mathrm{T}}K + Q = 0.$$

2.52.3 Theorem

If (A, B) is controllable and (C, A) is observable then CARE has a unique positive definite symmetric solution.